Stochastic limit approximation for rapidly decaying systems

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The stochastic limit approximation method for "rapid" decay is presented, where the damping rate γ is comparable to the system frequency Ω, i.e., $\gamma \sim \Omega$, whereas the usual stochastic limit approximation is applied only to the weak damping situation $\gamma \ll \Omega$. The key formulas for rapid decay are very similar to those for weak damping, but the dynamics are quite different. From a microscopic Hamiltonian, the spin-boson model, a Bloch equation containing two independent time scales is derived. This is a useful method to extract the minimal dissipative dynamics at high temperature $k_B T \gg \hbar\Omega$ and the master equations obtained are of the Lindblad form unlike that of Caldeira and Leggett. The validity of the method is confirmed by comparing the master equation derived through this method with the exact one.

One of the most attractive issues in fundamental physics is to understand dissipative dynamics from a microscopic point of view. This is not a trivial question since quantum theory is designed for closed systems and has time-reversal symmetry. A standard approach to this problem is to deal with the (dissipative) system of interest together with its surroundings, the "environment," which has infinite degrees of freedom [1–4]. The total system—"system" + "environment"—is then a closed system and is treated in a quantum-mechanical way. Caldeira and Leggett succeeded in formulating the "quantum Brownian motion" by making use of the path-integral method [5–7], and recent developments in technology have revealed the validity of this approach in the field of quantum optics [8].

One of the powerful methods of deriving dissipative dynamics from the total system approach is the stochastic limit approximation method established by Accardi *et al.* [9,10]. In quantum optics, for example, one is often interested in the decay of an excited atom in the radiation field. The total Hamiltonian reads

$$H_{\text{tot}} = H_S + \lambda V + H_B, \tag{1}$$

H_S and H_B being the Hamiltonians of the atom and the radiation field, respectively. Since the electromagnetic interaction λV is weak ($\lambda \ll 1$), the weak-coupling limit $\lambda \to 0$ gives a good approximation. Simultaneously making the coarse graining in time $t = \tau/\lambda^2$ ($\lambda \to 0$ with τ fixed), the stochastic limit approximation extracts the minimal (but notable) dissipative dynamics [11–13].

However, the usual stochastic limit approximation applied to the Hamiltonian (1) gives a rather restricted dynamics, i.e., *weak damping*. The lowest-order contribution of the interaction to the damping rate γ is proportional to λ^2, and it is much smaller than the characteristic

frequency of the system Ω, i.e., $\gamma \ll \Omega$, which is the case of quantum optics. For faster decay $\gamma \sim \Omega$, which we call *rapid decay*, a separate treatment is needed.

The rapidly decaying dynamics is also of physical interest, because it includes the case of quantum Brownian motion [2]. It contains a richer variety of transient dynamics in contrast to the situation of weak damping, where frequent oscillations of the system allow the "rotating-wave approximation," which simplifies the dynamics considerably [14].

In this paper, we explore the stochastic limit approximation for rapid decay. The total Hamiltonian is [2,12]

$$H_{\text{tot}} = \lambda^2 H_S + \lambda V + H_B. \tag{2}$$

The system frequency Ω is, in turn, of order $O(\lambda^2)$, and the resultant dissipative dynamics exhibits rapid decay $\gamma \sim \Omega$. It is shown that the key formulas are different from but similar to those obtained in the usual stochastic limit approximation for weak damping. This brings us with a Bloch equation with two independent time scales from a microscopic Hamiltonian, i.e., the spin-boson model. Furthermore, the master equations derived through this method are shown to be of the Lindblad form [15], which ensures the conservation and positivity of probability, unlike [2,16,17] that of Caldeira and Leggett [5].

For the sake of simplicity, we shall focus on the following spin-boson model [18]. The extension of the spin system to a general N-level system is straightforward. The total Hamiltonian is given by Eq. (2) with

$$H_S = \frac{\varepsilon}{2}\sigma_z + \frac{\Delta}{2}\sigma_x, \quad H_B = \int dk\, \hbar\omega_k a_k^\dagger a_k, \tag{3a}$$

$$V = i\hbar\sigma_z \int dk \left(g_k a_k - g_k^* a_k^\dagger \right). \tag{3b}$$

The spin system H_S, which has two energy eigenstates

$$H_S|\pm\rangle = \pm\frac{1}{2}\hbar\Omega_0|\pm\rangle, \quad \hbar\Omega_0 = \sqrt{\varepsilon^2 + \Delta^2}, \qquad (4)$$

interacts with the boson system H_B through the linear coupling interaction V. The particular choice of the coupling here is not essential except the linearity; g_k is the coupling constant for the boson of mode k, and throughout this paper it is assumed, as usual, that, before time $t = 0$, the two systems are uncorrelated and the boson system is in the thermal equilibrium state at temperature T. The initial state of the total system $\rho_{\rm tot}(0)$ is thus given by

$$\rho_{\rm tot}(0) = \rho_S \otimes \rho_B, \qquad (5a)$$

$$\rho_B = \frac{1}{Z}e^{-H_B/k_B T}, \quad Z = {\rm tr}_B\, e^{-H_B/k_B T}, \qquad (5b)$$

where k_B is the Boltzmann constant. This model was discussed by the usual stochastic limit approximation in Ref. [10] but with the total Hamiltonian (1).

With $H_0 = \lambda^2 H_S + H_B$ being the free part of the Hamiltonian, the time-evolution operator in the interaction picture, $U_I(t)$, satisfies the Schrödinger equation

$$\frac{d}{dt}U_I(t) = -\frac{i}{\hbar}\lambda V_I(t)U_I(t), \quad U_I(0) = 1, \qquad (6a)$$

$$V_I(t) = e^{iH_0 t/\hbar}Ve^{-iH_0 t/\hbar}$$
$$= i\hbar\sum_\alpha \left(D_\alpha A_\alpha(t) - D_\alpha^\dagger A_\alpha^\dagger(t)\right). \qquad (6b)$$

Here D_α ($\alpha = \{+, -, 0\}$) are the basic operators of the spin system,

$$D_\pm = |\pm\rangle\langle\mp|, \quad D_0 = |+\rangle\langle+| - |-\rangle\langle-|, \qquad (7)$$

and $A_\alpha(t)$ the operators of the Bose field,

$$A_\alpha(t) = C_\alpha \int dk\, g_k a_k e^{-i(\omega_k - \lambda^2\omega_\alpha)t}, \qquad (8)$$

with $C_+ = \langle+|\sigma_z|-\rangle = \Delta/\hbar\Omega_0 = C_-^*$, $C_0 = \langle+|\sigma_z|+\rangle = -\langle-|\sigma_z|-\rangle = \varepsilon/\hbar\Omega_0$, $\omega_\pm = \pm\Omega_0$, and $\omega_0 = 0$. Coarse graining in time is made in the Schrödinger equation (6): Changing the time scale from the microscopic time t to the macroscopic one $\tau = \lambda^2 t$,

$$\frac{d}{d\tau}U_I(\tau/\lambda^2)$$
$$= \sum_\alpha \left(D_\alpha\frac{1}{\lambda}A_\alpha(\tau/\lambda^2) - D_\alpha^\dagger\frac{1}{\lambda}A_\alpha^\dagger(\tau/\lambda^2)\right)U_I(\tau/\lambda^2), \qquad (9)$$

we take the stochastic limit $\lambda \to 0$ and obtain

$$\frac{d}{d\tau}\mathcal{U}_I(\tau) = \sum_\alpha \left(D_\alpha b_\alpha(\tau) - D_\alpha^\dagger b_\alpha^\dagger(\tau)\right)\mathcal{U}_I(\tau). \qquad (10)$$

Note that at the present stage, Eq. (10) should not be regarded as mathematically fully justified. A more rigorous analysis requires a prescription to handle normal ordering [9,10]. We shall see later that a consistent procedure of normal ordering can be obtained for the Heisenberg equations of the dressed spin system operators.

Let us compute the correlation functions of the operators of the Bose field $b_\alpha(\tau)$ in the thermal state ρ_B. These operators play the role of the "quantum noises" in the macroscopic time scale. First notice the basic formula

$$\lim_{\lambda\to 0}\int_{-\infty}^\infty d\tau\, F(\tau)\int_{-\infty}^\infty \frac{d\omega}{2\pi}G(\omega)\frac{1}{\lambda^2}e^{-i(\omega-\lambda^2\tilde\omega)\tau/\lambda^2}$$
$$= \lim_{\lambda\to 0}\int_{-\infty}^\infty dt\, F(\lambda^2 t)\hat{G}(t)e^{i\lambda^2\tilde\omega t}$$
$$= F(0)\lim_{\lambda\to 0}G(\lambda^2\tilde\omega). \qquad (11)$$

$\hat{G}(t)$ is the Fourier transform of $G(\omega)$. The correlation functions in the stochastic limit read

$$\langle b_\alpha(\tau)b_{\alpha'}^\dagger(\tau')\rangle_B$$
$$= \lim_{\lambda\to 0}\Gamma_{\alpha\alpha'}^+(\lambda^2\omega_{\alpha\alpha'})e^{i(\omega_\alpha-\omega_{\alpha'})\tau}\delta(\tau-\tau'), \qquad (12a)$$

$$\langle b_\alpha^\dagger(\tau)b_{\alpha'}(\tau')\rangle_B$$
$$= \lim_{\lambda\to 0}\left(\Gamma_{\alpha\alpha'}^-(\lambda^2\omega_{\alpha\alpha'})\right)^* e^{-i(\omega_\alpha-\omega_{\alpha'})\tau}\delta(\tau-\tau') \qquad (12b)$$

with the spectral functions

$$\Gamma_{\alpha\alpha'}^+(\omega) = \left(1 + N(\omega)\right)\Gamma_{\alpha\alpha'}(\omega), \qquad (13a)$$

$$\Gamma_{\alpha\alpha'}^-(\omega) = N(\omega)\Gamma_{\alpha\alpha'}(\omega), \qquad (13b)$$

$$\Gamma_{\alpha\alpha'}(\omega) = C_\alpha C_{\alpha'}^* \Gamma(\omega) = 2\pi C_\alpha C_{\alpha'}^* \int dk\,|g_k|^2\delta(\omega_k - \omega), \qquad (13c)$$

the Bose-Einstein distribution function

$$N(\omega) = \frac{1}{e^{\hbar\omega/k_B T} - 1}, \qquad (14)$$

and the shorthand notation $\omega_{\alpha\alpha'} = (\omega_\alpha + \omega_{\alpha'})/2$. The correlation time of the operators of the Bose field $b_\alpha(\tau)$ is negligibly small in the macroscopic time, which makes it reasonable to call them quantum noises.

The factor $\lim_{\lambda\to 0}\Gamma_{\alpha\alpha'}^\pm(\lambda^2\omega_{\alpha\alpha'})$ is sensitive to the form of the spectral function $\Gamma_{\alpha\alpha'}(\omega)$ for small positive ω. [For $\omega < 0$, $\Gamma_{\alpha\alpha'}(\omega) = 0$ by the definition in Eq. (13c).] Noting that $N(\omega) \sim k_B T/\hbar\omega$ for $\omega \sim 0$, one sees that the

only spectral functions that are linear (Ohmic) for small positive ω,

$$\Gamma_{\alpha\alpha'}(\omega) \sim \eta_{\alpha\alpha'}\omega = C_\alpha C_{\alpha'}^*\eta\omega, \tag{15}$$

yield nontrivial values for $\lim_{\lambda\to 0}\Gamma_{\alpha\alpha'}^\pm(\lambda^2\omega_{\alpha\alpha'})$. For $\omega_{\alpha\alpha'} > 0$, the zeroth order of the expansion in $\lambda^2\hbar\omega_{\alpha\alpha'}/k_BT$,

$$\Gamma_{\alpha\alpha'}^\pm(\lambda^2\omega_{\alpha\alpha'})/(k_BT/\hbar) = \eta_{\alpha\alpha'} + O(\lambda^2\hbar\omega_{\alpha\alpha'}/k_BT), \tag{16}$$

survives in the stochastic limit $\lambda \to 0$, so that

$$\lim_{\lambda\to 0}\Gamma_{\alpha\alpha'}^\pm(\lambda^2\omega_{\alpha\alpha'}) = \begin{cases} \dfrac{k_BT}{\hbar}\eta_{\alpha\alpha'} & (\omega_{\alpha\alpha'} > 0) \\[2mm] \dfrac{k_BT}{2\hbar}\eta_{\alpha\alpha'} & (\omega_{\alpha\alpha'} = 0) \\[2mm] 0 & (\omega_{\alpha\alpha'} < 0). \end{cases} \tag{17}$$

The case $\omega_{\alpha\alpha'} = 0$ is treated separately since $\Gamma_{\alpha\alpha'}^\pm(\omega)$ is discontinuous at $\omega = 0$, and $\Gamma_{\alpha\alpha'}^\pm(0)$ is defined here by $\Gamma_{\alpha\alpha'}^\pm(0) = [\Gamma_{\alpha\alpha'}^\pm(0^+) + \Gamma_{\alpha\alpha'}^\pm(0^-)]/2$ for convention.

The correlation functions (12) are similar to their counterparts in the usual treatment for weak damping [9,10], which are reproduced by the replacement $\lambda^2\omega_\alpha \to \omega_\alpha$ in Eqs. (12). There is, however, a significant difference between them: In the usual stochastic limit for weak damping, the rapidly oscillating factor $e^{i(\omega_\alpha-\omega_{\alpha'})\tau/\lambda^2}$ extracts the $\omega_\alpha = \omega_{\alpha'}$ contribution only and makes the "rotating-wave approximation" exact [14], while in the present case, the rotating-wave approximation is not applicable in general.

Now let us derive the Heisenberg equations for the dressed spin system operators in the bath,

$$\mathcal{D}_\alpha(\tau) = \mathrm{tr}_B\left(\rho_B\mathcal{U}_I^\dagger(\tau)D_\alpha e^{i\omega_\alpha\tau}\mathcal{U}_I(\tau)\right), \tag{18}$$

from the Schrödinger equation (10). The above set of equations yields the Bloch equation and is immediately translated into the master equation. Differentiating both sides of Eq. (18) with respect to τ, and applying the Schrödinger equation (10) to the time derivative of $\mathcal{U}_I(\tau)$, we will obtain the equations for $\mathcal{D}_\alpha(\tau)$ after computing the partial trace with the thermal state ρ_B in Eq. (18). Since we are working in the operator formalism, we rely upon the techniques of Thermo Field Dynamics (TFD) [19] for the evaluation of the trace with the thermal state. In the language of TFD, the thermal state is represented by "thermal vacuum" $|\theta\rangle$. It is annihilated by the annihilation operator ξ_k and its tilde conjugate $\tilde\xi_k$, which are related to the operator a_k by

$$a_k = \sqrt{1 + N(\omega_k)}\xi_k + \sqrt{N(\omega_k)}\tilde\xi_k^\dagger. \tag{19}$$

Thus the key formulas for the derivation of the Heisenberg equations for $\mathcal{D}_\alpha(\tau)$ are the commutation relations

between $\mathcal{U}_I(\tau)$ and the operators $\chi_\alpha(\tau)$, $\tilde\chi_\alpha(\tau)$, defined as the stochastic limit of the operators $\Xi_\alpha(\tau/\lambda^2)/\lambda$ and $\tilde\Xi_\alpha(\tau/\lambda^2)/\lambda$:

$$\Xi_\alpha(t) = C_\alpha\int dk\sqrt{1 + N(\omega_k)}g_k\xi_k e^{-i(\omega_k-\lambda^2\omega_\alpha)t}, \tag{20a}$$

$$\tilde\Xi_\alpha(t) = C_\alpha^*\int dk\sqrt{N(\omega_k)}g_k^*\tilde\xi_k e^{i(\omega_k-\lambda^2\omega_\alpha)t}. \tag{20b}$$

Notice that they are related to $b_\alpha(\tau)$ by $b_\alpha(\tau) = \chi_\alpha(\tau) + \tilde\chi_\alpha^\dagger(\tau)$.

The commutation relations enable us to make normal ordering and to evaluate the partial trace. They are calculated in a way similar to that in the usual stochastic limit for weak damping [9,10]: Observe the commutation relations between $\Xi_\alpha(\tau/\lambda^2)/\lambda$, $\tilde\Xi_\alpha(\tau/\lambda^2)/\lambda$ and the iterative solution

$$U_I(\tau/\lambda^2) = 1 - \frac{i}{\hbar}\int_0^\tau d\tau'\frac{1}{\lambda}V_I(\tau'/\lambda^2)U_I(\tau'/\lambda^2), \tag{21}$$

and then take the stochastic limit. The key commutation relations are obtained as

$$[\chi_\alpha(\tau),\mathcal{U}_I(\tau)] = -\sum_{\alpha'}\lim_{\lambda\to 0}\left(i\Sigma_{\alpha\alpha'}^+(\lambda^2\omega_{\alpha\alpha'})\right) \\ \times e^{i(\omega_\alpha-\omega_{\alpha'})\tau}D_{\alpha'}^\dagger\mathcal{U}_I(\tau), \tag{22a}$$

$$[\tilde\chi_\alpha(\tau),\mathcal{U}_I(\tau)] = \sum_{\alpha'}\lim_{\lambda\to 0}\left(i\Sigma_{\alpha\alpha'}^-(\lambda^2\omega_{\alpha\alpha'})\right)^* \\ \times e^{-i(\omega_\alpha-\omega_{\alpha'})\tau}D_{\alpha'}\mathcal{U}_I(\tau). \tag{22b}$$

Here $\Sigma_{\alpha\alpha'}^\pm(\omega)$ is the self-energy function, given by

$$\Sigma_{\alpha\alpha'}^\pm(\omega) = -i\int_0^\infty dt\int_0^\infty\frac{d\omega'}{2\pi}\Gamma_{\alpha\alpha'}^\pm(\omega')e^{-i(\omega'-\omega)t} \tag{23a}$$

$$= \int_0^\infty\frac{d\omega'}{2\pi}\Gamma_{\alpha\alpha'}^\pm(\omega')\frac{1}{\omega-\omega'+i0^+} \tag{23b}$$

$$= \Delta_{\alpha\alpha'}^\pm(\omega) - \frac{i}{2}\Gamma_{\alpha\alpha'}^\pm(\omega), \tag{23c}$$

whose imaginary part $\Gamma_{\alpha\alpha'}^\pm(\omega)$ will give the decay rate, and real part

$$\Delta_{\alpha\alpha'}^\pm(\omega) = \mathcal{P}\int_0^\infty\frac{d\omega'}{2\pi}\Gamma_{\alpha\alpha'}^\pm(\omega')\frac{1}{\omega-\omega'} \tag{23d}$$

will contribute to the energy shift in the following Bloch equation or in the master equation. The counterparts in the usual stochastic limit for weak damping [9,10] are again reproduced from Eqs. (22) by the same replacement $\lambda^2\omega_\alpha \to \omega_\alpha$ as that for the correlation functions (12).

By making use of the commutation relations (22), the Bloch equation for rapidly decaying spin system is obtained:

$$\frac{d}{d\tau}\begin{pmatrix}\mathcal{D}_+(\tau)\\\mathcal{D}_0(\tau)\\\mathcal{D}_-(\tau)\end{pmatrix}=\begin{pmatrix}-(\tilde{\Delta}^2+2\tilde{\varepsilon}^2)\gamma^\theta/2+i\Omega_0 & \tilde{\varepsilon}\tilde{\Delta}\gamma^\theta/2 & \tilde{\Delta}^2\gamma^\theta/2\\ \tilde{\varepsilon}\tilde{\Delta}\gamma^\theta & -\tilde{\Delta}^2\gamma^\theta & \tilde{\varepsilon}\tilde{\Delta}\gamma^\theta\\ \tilde{\Delta}^2\gamma^\theta/2 & \tilde{\varepsilon}\tilde{\Delta}\gamma^\theta/2 & -(\tilde{\Delta}^2+2\tilde{\varepsilon}^2)\gamma^\theta/2-i\Omega_0\end{pmatrix}\begin{pmatrix}\mathcal{D}_+(\tau)\\\mathcal{D}_0(\tau)\\\mathcal{D}_-(\tau)\end{pmatrix}, \tag{24}$$

where $\tilde{\varepsilon}=\varepsilon/\hbar\Omega_0$, $\tilde{\Delta}=\Delta/\hbar\Omega_0$, and

$$\gamma^\theta=2\eta k_B T/\hbar. \tag{25}$$

Through the relations $\langle\pm|\rho_S(\tau)|\mp\rangle=\mathrm{tr}_S[\rho_S\mathcal{D}_\mp(\tau)]$ and $\langle\pm|\rho_S(\tau)|\pm\rangle=\{1\pm\mathrm{tr}_S[\rho_S\mathcal{D}_0(\tau)]\}/2$, this gives the master equation for the density operator of the spin system, $\rho_S(\tau)=\mathrm{tr}_B\,\rho_{\mathrm{tot}}(\tau)$,

$$\frac{d}{d\tau}\rho_S(\tau)=-\frac{i}{\hbar}[H_S,\rho_S(\tau)]-\frac{\gamma^\theta}{4}[\sigma_z,[\sigma_z,\rho_S(\tau)]]. \tag{26}$$

Note that, for the model considered here, the energy shift disappears in the stochastic limit.

The key formulas for weak damping and those for rapid decay are quite similar: The formulas for weak damping are reproduced by the formal replacement $\lambda^2\omega_\alpha\to\omega_\alpha$ in those for rapid decay in Eqs. (12) and (22). However, in the case of weak damping where the total Hamiltonian is given by Eqs. (1) and (3), the Bloch equation is much simpler than that for rapid decay (24). The matrix in Eq. (24) for weak damping is diagonal (or decoupled), and furthermore, the damping coefficients of $\mathcal{D}_\pm(\tau)$ and $\mathcal{D}_0(\tau)$ are given by $\gamma_D^\theta=\tilde{\Delta}^2\gamma^\theta/2$ and $\gamma_R^\theta=\tilde{\Delta}^2\gamma^\theta$, respectively, but with $\gamma^\theta=\Gamma(\Omega_0)\coth(\hbar\Omega_0/2k_BT)$ instead of that for rapid decay (25) [10], i.e., the relation $\tau_D=2\tau_R$ always holds between the decoherence time $\tau_D=(\gamma_D^\theta)^{-1}$ and the thermal relaxation time $\tau_R=(\gamma_R^\theta)^{-1}$, while it does not always hold in spin-relaxation experiments. The phenomenological Bloch equation [20] is given two independent time scales τ_R and τ_D. The simplicity of the weak damping formulas is due to the rotating-wave approximation, arising from the frequent oscillations of the spin system $e^{i(\omega_\alpha-\omega_{\alpha'})\tau/\lambda^2}$, which suppresses many terms in the equations [14]. The Bloch equation (24) for rapidly decaying systems, on the other hand, contains a richer variety. Between the two decay constants, which are the real parts of the eigenvalues of the matrix in Eq. (24), there is no trivial relation as that for weak damping. Even more, there is a case, depending on the parameters, where three decay constants (three real eigenvalues) exist. Details of these features will be reported in elsewhere.

It is easy to show that the master equation (26) has the unique thermal equilibrium state ρ_{eq}, which is proportional to 1: $\rho_S(\tau)\to\rho_{\mathrm{eq}}\propto 1$ as $\tau\to\infty$. ρ_{eq} is nothing but the thermal state at infinitely high temperature. This is because the master equation (26) is valid for the situation $\lambda^2\hbar\Omega_0/k_BT\ll 1$. Remember the expansion (16). The stochastic limit approximation is a

method to extract the minimal dissipative dynamics from the full of it. It picks only the processes where one boson is emitted or absorbed, and neglects the higher-order contributions [13]. As for the stochastic limit approximation for rapid decay, one can say, in addition, that it extracts the dissipative dynamics *at a much higher temperature* than the characteristic energy scales of the system.

The master equation (26) derived through the stochastic limit approximation for rapid decay is worthy of note. It is of the Lindblad form, which ensures the conservation and positivity of probability [15]. For the case of weak damping, many authors derived master equations of the Lindblad form, while for the case of rapid decay, much care is required [2,17]. The master equation derived by Caldeira and Leggett [5], for example, is not of the Lindblad form [2,16,17]. Such master equations lead, in certain cases, to unphysical results [17].

It is interesting to look at the master equation for the model whose total Hamiltonian is given by Eq. (2) with

$$H_S=\frac{1}{2M}p^2+\frac{1}{2}M\Omega_0^2 x^2, \tag{27a}$$

$$V=-i\hbar\sqrt{\frac{M\Omega_0}{\hbar}}x\int dk\left(g_k a_k-g_k^* a_k^\dagger\right), \tag{27b}$$

and the same H_B as in Eqs. (3a). The master equation derived through the stochastic limit approximation for rapid decay illustrated here reads

$$\frac{d}{d\tau}\rho_S(\tau)=-\frac{i}{\hbar}[H_S^\theta,\rho_S(\tau)]-\frac{M\Omega_0\eta k_BT}{2\hbar^2}[x,[x,\rho_S(\tau)]], \tag{28a}$$

where H_S^θ is the renormalized Hamiltonian with the renormalized frequency

$$\Omega_R^2=\Omega_0^2-2\Omega_0\int_0^\infty\frac{d\omega}{2\pi}\frac{\Gamma(\omega)}{\omega}. \tag{28b}$$

The master equation (28a) is of the Lindblad form since it lacks the term $-(i\Omega_0\eta/4\hbar)[x,\{p,\rho_S(\tau)\}]$ which is contained in the one derived by Caldeira and Leggett [5]. This term may be neglected, when compared to the last term of the master equation (28a) if the temperature T is high enough. The same situation is found in the spin-boson model. The master equation derived by Munro and Gardiner in Ref. [17], which is again not of the Lindblad form, is reduced to the Lindblad form master equation (26) in the high-temperature limit.

The model (27) treated above is exactly solvable [4,6,7,14], and it is possible to write down an exact master equation for the reduced density operator [7]. Let us finally confirm the validity of the master equation (28) derived through the method presented here by comparing it with the exact one. The master equation, which is exact for an arbitrary λ, is obtained through the method sketched in the Appendix and reads

$$i\hbar \frac{d}{d\tau}\rho_S(\tau) = [H_S^\theta(\tau), \rho_S(\tau)] - iD_{xx}(\tau)[x, [x, \rho_S(\tau)]]$$
$$- 2iD_{xp}(\tau)[x, [p, \rho_S(\tau)]] + \Gamma_{xp}(\tau)[x, \{p, \rho_S(\tau)\}], \quad (29a)$$

where

$$H_S^\theta(\tau) = \frac{1}{2M}p^2 + \frac{1}{2}M\Omega_R^2(\tau)x^2. \quad (29b)$$

The time-dependent coefficients $\Omega_R(\tau)$, $D_{xx}(\tau)$, $D_{xp}(\tau)$, and $\Gamma_{xp}(\tau)$ are given by Eqs. (A6). They are reduced in the stochastic limit $\lambda \to 0$, as shown in the Appendix, to

$$\Omega_R^2(\tau) \to \Omega_R^2, \quad D_{xx}(\tau) \to \frac{M\Omega_0\eta k_B T}{2\hbar},$$
$$D_{xp}(\tau) \to 0, \quad \Gamma_{xp}(\tau) \to 0, \quad (30)$$

and the master equation (28) is thus reproduced from the exact one (29), which shows the consistency of our method.

We have discussed the stochastic limit approximation for rapid decay and extended its applicability. Although some mathematical issues, e.g., the question of the convergence of the operators, remain unaddressed, the framework presented here is useful for practical calculations.

APPENDIX:

Here we briefly illustrate a derivation of the exact master equation (29) for the model (2) with H_S, V in Eqs. (27), and H_B in Eqs. (3a), and show that the stochastic limit of it reproduces the master equation (28).

The derivation of the exact master equation (29) is found in Ref. [7], which is based on the influence functional approach proposed by Feynman and Vernon [3]. The path integration of the time-evolution kernel for the reduced density matrix in the position representation, $J(x_f, x'_f, \tau_f|x_i, x'_i, \tau_i)$ [with which

the reduced density matrix is given by $\langle x|\rho_S(\tau)|x'\rangle = \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dx'_0 \, J(x, x', \tau|x_0, x'_0, 0)\langle x_0|\rho_S|x'_0\rangle]$, is carried out without any approximation to give [6,7]

$$J(x_f, x'_f, \tau|x_i, x'_i, 0) = \frac{1}{2\pi\hbar}|\Lambda_{if}(\tau)|$$
$$\times \exp\left[\sum_{k,l}\left(\frac{i}{\hbar}\Delta_k\Lambda_{kl}(\tau)\Sigma_l - \frac{1}{\hbar}\Delta_k\Theta_{kl}(\tau)\Delta_l\right)\right], \quad (A1)$$

where $\Sigma_k = (x_k + x'_k)/2$, $\Delta_k = x_k - x'_k$ ($k = \{f, i\}$),

$$\Lambda_{ff}(\tau) = \Lambda_{ii}(\tau) = M\frac{d}{d\tau}\ln|F(\tau)|, \quad (A2a)$$

$$\Lambda_{fi}(\tau) = \lambda^2 MF(\tau)\frac{d^2}{d\tau^2}\ln|F(\tau)|, \quad \Lambda_{if}(\tau) = -\frac{M}{\lambda^2 F(\tau)}, \quad (A2b)$$

and

$$\Theta_{kl}(\tau) = \frac{1}{2}\int_0^\tau d\tau_1 \int_0^\tau d\tau_2 \, c_k(\tau, \tau_1)\nu(\tau_1 - \tau_2)c_l(\tau, \tau_2), \quad (A3a)$$

$$c_f(\tau, \tau') = \lambda^2\dot{F}(\tau') - \left(\frac{d}{d\tau}\ln|F(\tau)|\right)\lambda^2 F(\tau'), \quad (A3b)$$

$$c_i(\tau, \tau') = \frac{F(\tau')}{F(\tau)}, \quad (A3c)$$

$$\nu(\tau) = M\Omega_0\int_0^\infty \frac{d\tilde{\omega}}{2\pi}\Gamma(\lambda^2\tilde{\omega})\coth\frac{\lambda^2\hbar\tilde{\omega}}{2k_B T}\cos\tilde{\omega}\tau. \quad (A3d)$$

$F(\tau)$ is the solution of the integrodifferential equation

$$\ddot{F}(\tau) + \Omega_0^2 F(\tau) + \frac{2}{M}\int_0^\tau d\tau'\mu(\tau - \tau')F(\tau') = 0, \quad (A4a)$$

$$\mu(\tau) = -M\Omega_0\int_0^\infty \frac{d\tilde{\omega}}{2\pi}\Gamma(\lambda^2\tilde{\omega})\sin\tilde{\omega}\tau \quad (A4b)$$

with the initial condition $F(0) = 0$, $\dot{F}(0) = 1$, and is given by an inverse Laplace transformation

$$\lambda^2 F(\tau) = \int_C \frac{d\tilde{s}}{2\pi i}\frac{1}{\tilde{s}^2 + \Omega_0^2 + 2\hat{\mu}(\lambda^2\tilde{s})/M}e^{\tilde{s}\tau}, \quad (A5a)$$

$$\hat{\mu}(s) = -M\Omega_0\int_0^\infty \frac{d\omega}{2\pi}\Gamma(\omega)\frac{\omega}{s^2 + \omega^2}. \quad (A5b)$$

C is the so-called Bromwich path. Note that time t has already been rescaled to $\tau = \lambda^2 t$ (and accordingly, the integration variable ω to $\tilde{\omega} = \omega/\lambda^2$), but λ is so far finite, and the solution (A1) is exact for an arbitrary λ.

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The exact master equation (29) is then derived by differentiating the kernel (A1) with respect to time τ, replacing Δ_i and Σ_i with linear combinations of $\partial/\partial\Delta_f$, $\partial/\partial\Sigma_f$, Δ_f, and Σ_f [7], multiplying by the initial density $\langle x_i|\rho_S|x_i'\rangle$, and integrating over the initial coordinates x_i, x_i'. The coefficients of the master equation (29) are given by

$$\Omega_R^2(\tau) = \frac{1}{M}\Lambda_{ff}\frac{d}{d\tau}\ln\left|\frac{\Lambda_{fi}}{\Lambda_{ff}}\right|, \tag{A6a}$$

$$D_{xx}(\tau) = 4\Gamma_{xp}(\tau)\left(\Theta_{ff} - \frac{\Theta_{fi}}{\Lambda_{if}}\Lambda_{ff}\right)$$
$$+ \dot{\Theta}_{ff} - \frac{2\dot{\Theta}_{fi}}{\Lambda_{if}}\Lambda_{ff} + \frac{\dot{\Theta}_{ii}}{\Lambda_{if}^2}\Lambda_{ff}^2, \tag{A6b}$$

$$D_{xp}(\tau) = \frac{2\Theta_{fi}}{\Lambda_{if}}\Gamma_{xp}(\tau) + \frac{1}{M}\left(\Theta_{ff} - \frac{\Theta_{fi}}{\Lambda_{if}}\Lambda_{ff}\right)$$
$$+ \frac{\dot{\Theta}_{fi}}{\Lambda_{if}} - \frac{\dot{\Theta}_{ii}}{\Lambda_{if}^2}\Lambda_{ff}, \tag{A6c}$$

and

$$\Gamma_{xp}(\tau) = -\frac{1}{2}\left(\frac{d}{d\tau}\ln|\Lambda_{fi}| + \frac{1}{M}\Lambda_{ff}\right). \tag{A6d}$$

Now we observe that the master equation (28) is reproduced from the exact one (29) in the stochastic limit $\lambda \to 0$. First notice that the function $\lambda^2 F(\tau)$ is evaluated in the limit as

$$\lambda^2 F(\tau) \to \int_C \frac{d\tilde{s}}{2\pi i}\frac{1}{\tilde{s}^2 + \Omega_R^2}e^{\tilde{s}\tau} = \frac{1}{\Omega_R}\sin\Omega_R\tau \tag{A7}$$

with the renormalized frequency Ω_R given in Eq. (28b). $\Lambda_{kl}(\tau)$ are hence given by

$$\Lambda_{ff}(\tau) = \Lambda_{ii}(\tau) \to M\Omega_R\cot\Omega_R\tau, \tag{A8a}$$

$$\Lambda_{fi}(\tau),\ \Lambda_{if}(\tau) \to -\frac{M\Omega_R}{\sin\Omega_R\tau}, \tag{A8b}$$

and $\Theta_{kl}(\tau)$ by

$$\Theta_{ff}(\tau),\ \Theta_{ii}(\tau) \to \frac{M\Omega_0\eta k_B T}{\hbar}\frac{\Omega_R\tau - \sin\Omega_R\tau\cos\Omega_R\tau}{4\Omega_R\sin^2\Omega_R\tau}, \tag{A9a}$$

$$\Theta_{fi}(\tau) = \Theta_{if}(\tau)$$
$$\to -\frac{M\Omega_0\eta k_B T}{\hbar}\frac{\Omega_R\tau\cos\Omega_R\tau - \sin\Omega_R\tau}{4\Omega_R\sin^2\Omega_R\tau}. \tag{A9b}$$

It is easy to see that the substitution (A8) and (A9) brings us with those constants given in Eqs. (30), and the exact master equation (29) is reduced to the master equation (28).

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